REGULAR COGRAPH IS DETERMINED BY ITS SPECTRUM

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Abstract. A graph G is said to be determined by its spectrum if there does not exist other non-isomorphic graph H such that H and G have the same spectrum. In this paper, we give a complete spectral characterization of regular graphs which are cographs, providing closed formulas for its Laplacian eigenvalues and we prove they are determined by their spectrum.

keywords: Laplacian eigenvalues, cographs, graphs L-DS

1. Introduction

Throughout this article, all graphs are assumed to be finite, undirected, and without loops or multiple edges. We first set some notation and terminology. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, denote by $N(v)$ the *open neighborhood* of v, that is, $\{w|\{v,w\} \in E\}$ and by $N[v] := N(v) \cup \{v\}$ the *closed neighborhood* of v. Two vertices $u, v \in V(G)$ are *duplicate* if $N(u) = N(v)$ and *coduplicate* if $N[u] = N[v]$. If $|V| = n$, the adjacency matrix $A = [a_{ij}]$ is the $n \times n$ matrix of zeros and ones such that $a_{ij} = 1$ if and only if v_i is adjacent to v_j.

The degree sequence of a graph G of order n, is the sequence $\delta(v_1), \ldots, \delta(v_n)$, where $\delta(v_i)$ is the degree of vertex v_i. Let $\delta(G)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined as $L(G) = \delta(G) - A(G)$. The A-eigenvalues and L-eigenvalues of G are the respective eigenvalues of $A(G)$ and $L(G)$, denoted by $Spect_A(G) = \{\lambda_1, \ldots, \lambda_n\}$ and $Spect_L(G) = \{\mu_n, \ldots \mu_2, \mu_1 = 0\}$.

Two graphs are said to be Laplacian cospectral (for short, L-cospectral), if they share the same Laplacian spectrum. A graph G is said to be determined by its Laplacian spectrum (for short, L-DS) if any other non-isomorphic graph has a different Laplacian spectrum.

The notion of a graph G to be DS is originally defined for the adjacency matrix of the graph G, but a natural extension of the problem is to find families of graphs that are determined by the spectrum in relation to other matrices. Finding families of non-DS graphs is a related relevant problem and there are many constructions in the literature [9, 10, 16, 20].

In this paper, we investigate the L-cospectrality in the class of regular graphs which are cographs. It is well known that cograph can be represented by rooted tree, and a lot of spectral properties about a cograph may be obtained from a tree that produces it, (see, for example [5, 6, 11, 12, 19]). In this way, we use a linear algorithm that locates its Laplacian eigenvalues for exploring spectral properties of this class of graphs. We give a complete spectral characterization of regular graphs which are cographs, providing closed formulas for its Laplacian eigenvalues and we prove they are L-DS.

Our paper is organized as follows. In Section 2, we provide definitions and known results needed for the development of our paper. In Section 3, for a regular cograph,

Date: May 2023.

we provide closed formulas for its Laplacian eigenvalues. Finally, in the last section, we prove that regular cographs are L-DS.

2. Preliminaries

2.1. **Cographs and cotrees.** In what follows, G denotes a graph with n vertices, while that $\overline{G}$ its complement. As usual, K_n, nK_1, C_n, P_n represent the complete graph, the edgeless graph, the cycle graph and the path graph of order n, respectively. Now, we recall the definitions of some operations with graphs that will be used. For this, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint graphs:

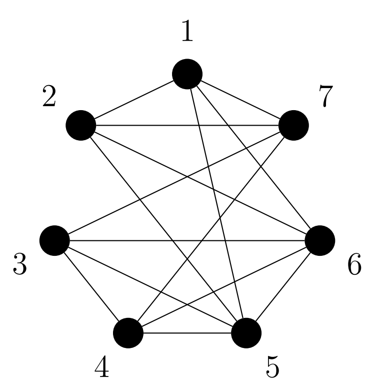
- The *union* of graphs G_1 and G_2 is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$.
- The *join* of graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

If G_1 and G_2 are graphs on n_1 and n_2 vertices respectvely, with eigenvalues $\mu_{n_1}(G_1) \geq \ldots \geq \mu_2(G_1) \geq \mu_1(G_1) = 0$ and $\mu_{n_2}(G_2) \geq \ldots \geq \mu_2(G_2) \geq \mu_1(G_2) = 0$, respectively, then the Laplacian eigenvalues of $G_1 \otimes G_2$ are given by $0, \mu_2(G_1) + n_2, \ldots, \mu_{n_1}(G_1) + n_2, \mu_2(G_2) + n_1, \ldots, \mu_{n_2}(G_2) + n_1, n_1 + n_2$. We note that for any graph G on n vertices, its largest Laplacian eigenvalue $\mu_n(G)$, satisfies $\mu_n(G) \leq n$, with equality holding if and only if G is a join of two graphs. Finally, if $\mu_i(G)$ is a Laplacian eigenvalue of G on n vertices then $n - \mu_i(G)$ is a Laplacian eigenvalue of $\overline{G}$.

A cograph is a simple graph which contains no path on four vertices an induced subgraph, namely it is a P_4-free graph. An equivalent definition (see [7]) is that cographs can be obtained recursively by using the graph operations of union and join. Other ways to define cographs can be viewed in [2, 18, 13].

Each cograph can be represented by a tree, called a *cotree* [6]. A cotree T_G of a cograph G is a rooted tree in which any interior vertex w is either of $\cup$-type (corresponds to union) or $\otimes$-type (corresponds to join). The leaves are typeless and represent the vertices of the cograph G. An interior vertex is said to be terminal, if it has no interior vertex as successor. We say that *depth* of the cotree is the number of edges of the longest path from the root to a leaf. To build a cotree for a connected cograph, we simply place a $\otimes$ at the tree's root, placing $\cup$ on interior vertices with odd depth, and placing $\otimes$ on interior vertices with even depth.

As an illustration, we give a simple example. The Figure 4 shows a cograph G and its cotree T_G with depth equals to 3.



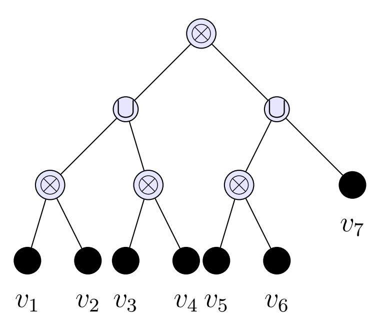


Figure 1. A cograph $G = ((v_1 \otimes v_2) \cup (v_3 \otimes v_4)) \otimes ((v_5 \otimes v_6) \cup v_7))$ and its cotree T_G.

2.2. **Diagonalization Algorithm.** An important tool presented in [13] was a linear algorithm for constructing a *diagonal* matrix congruent to $L(G)+xI_n$, where $L(G)$ is the Laplacian matrix of a cograph G, and x is an arbitrary scalar.

The algorithm's input is the cotree T_G and x. Each leaf v_i, $i=1,\dots,n$ have a value d_i that represents the diagonal element of $L(G)+xI_n$. It initializes all entries with $\delta(v_i)+x$, where $\delta(v_i)$ denotes the degree of vertex v_i. At each iteration, a pair $\{v_k,v_l\}$ of duplicate or coduplicate vertices with maximum depth is selected. Then they are processed, that is, assignments are given to d_k and d_l, such that either one or both rows (columns) are diagonalized. When a k row (column) corresponding to vertex v_k has been diagonalized then v_k is removed from the T_G, it means that d_k has a permanent final value. Then the algorithm moves to the cotree T_G-v_k. The algorithm is shown in Figure 2.

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INPUT: cotree T_G, scalar x
OUTPUT: diagonal matrix D = [d_1, d_2, ..., d_n] congruent to L(G) + xI_n

 Algorithm Diagonal (T_G, x)
   initialize d_i := δ(v_i) + x, for 1 ≤ i ≤ n
   while T_G has ≥ 2 leaves
      select a pair (v_k, v_l) (co)duplicate of maximum depth with parent w
      α ← d_k  β ← d_l
      if w = ⊗
         if α + β ≠ −2                          //subcase 1a
            d_l ← (αβ−1)/(α+β+2);   d_k ← α + β + 2;   T_G = T_G − v_k
         else if β = −1                          //subcase 1b
            d_l ← −1   d_k ← 0;   T_G = T_G − v_k
         else                                    //subcase 1c
            d_l ← −1   d_k ← (1 + β)^2;   T_G = T_G − v_k;   T_G = T_G − v_l
      else if w = ∪
         if α + β ≠ 0                            //subcase 2a
            d_l ← αβ/(α+β);   d_k ← α + β;   T_G = T_G − v_k
         else if β = 0                           //subcase 2b
            d_l ← 0;   d_k ← 0;   T_G = T_G − v_k
         else                                    //subcase 2c
            d_l ← β;   v_k ← −β;   T_G = T_G − v_k;   T_G = T_G − v_l
   end loop
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FIGURE 2. Diagonalization algorithm

The next result from [13] will be used throughout the paper.

Theorem 1. *Let G be a cograph and let $(d_v)_{v\in T_G}$ be the sequence produced by Diagonalize $(T_G,-x)$. Then the diagonal matrix $D=diag(d_v)_{v\in T_G}$ is congruent to $L(G)+xI_n$, so that the number of (positive - negative - zero) entries in $(d_v)_{v\in T_G}$ is equal to the number eigenvalues of $L(G)$ that are (greater than x - small than x - equal to x).*

The following two lemmas show that if a vertex $\otimes$-type or $\cup$-type, in the cotree, have leaves with the same value, then, we can use the following routines.

Lemma 1. *If $v_1, \dots, v_m$ have parent $w = \otimes$, each with the same diagonal value $y \neq -1$, then the algorithm performs $m-1$ iterations of* **subcase 1a** *assigning, during iteration j :*

$$d_k \leftarrow \frac{j+1}{j}(y+1) \quad d_l \leftarrow \frac{y-(j-1)}{j+1} \tag{1}$$

Lemma 2. *If $v_1, \dots, v_m$ have parent $w = \cup$, each with the same diagonal value $y \neq 0$, then the algorithm performs $m-1$ iterations of* **subcase 2a** *assigning, during iteration j :*

$$d_k \leftarrow \frac{j+1}{j}y \quad d_l \leftarrow \frac{y}{j+1} \tag{2}$$

Lemma 3. *Let G be a cograph with cotree T_G. Let $t_i \geq 2$ be leaves with degree $\delta(v_i)$ of an interior vertex w_i of T_G. Then*

i. *$\delta(v_i)$ is a Laplacian eigenvalue with multiplicity $t_i - 1$, if $w_i = \cup$-type.*
ii. *$\delta(v_i) + 1$ is a Laplacian eigenvalue with multiplicity $t_i - 1$, if $w_i = \otimes$-type.*

Proof. Let $t_i \geq 2$ be leaves with degree $\delta(v_i)$ of an interior vertex w_i of T_G. Initializing the Diagonalization with $x = -\delta(v_i)$ (respect. $x = -\delta(v_i) - 1$) for duplicate (respect. coduplicate) vertices it is easy to see that the algorithm enters to the **subcase 2b** (respect. **subcase 1b**) and assigns a permanent zero value. □

Lemma 4. *Let G be a connected cograph with cotree T_G. Then Diagonalization $(T_G, -x)$ assigned a permanent value zero in the last iteration if and only if $x = 0$ and the* **subcase 1a** *is executed.*

Proof. Let G be a connected cograph with cotree T_G. Let $x = 0$ be a Laplacian eigenvalue of G and consider the execution of Diagonalization $(T_G, -x)$. According Theorem 1, we must be a zero on the final diagonal. Doing a specious analysis on the algorithm the permanent value zero can be given in the following situations: by **subcase 1b**, if $\beta = -1$, by **subcase 2b**, if $\beta = 0$, and by **subcase 1a**, if $\alpha = 1/\beta$.

Since G is connected, the **subcase 2b** can not be executed in the last iteration. Now, if **subcase 1b** is executed in the last iteration, then the assignments given are $d_k = 0$ and $d_l = -1$. It means there is a Laplacian eigenvalue small than $x = 0$ according to Theorem 1, what is a contradiction. Therefore, follows the value zero assigned in the last iteration by **subcase 1a**. □

3. Regular graphs and cographs

In this section, we provide a complete spectral characterization of regular graphs which are cographs, including closed formulas for its Laplacian eigenvalues.

3.1. Balanced cotrees.

For positive integers $a_1, \dots, a_{s-1}, a_s$, the balanced cotree $T_G(a_1, \dots, a_{s-1}, 0|0, \dots, 0, a_s)$ of depth s corresponding to cograph G on $n = a_1 \dots a_{s-1}a_s$ vertices has a vertex $\otimes$ at the root, this vertex has exactly a_1 immediate $\cup$ interior vertices. Each $\cup$ at level 1 has exactly a_2 immediate $\otimes$ interior vertices, and so on. Notice that, this cotree only has leaves at the last level. It means that, the vertices at level $s-1$ have a_s immediate leaves. So, at level i, the cotree has $a_1 a_2 \cdots a_i$ vertices $\otimes$ if i is even and $\cup$ if i is odd, for $1 \leq i \leq s-1$. And, at the last level s, it has $a_1 a_2 \cdots a_s$ leaves. It is easy to check that cographs with balanced cotrees are regular graphs.

The Figure 3 shows the balanced cotree $T_G(2, 2, 0|0, 0, 2)$ with depth $s = 3$. For more details on balanced cotrees see [3, 4]. The complete graph K_n is a cograph with balanced cotree $T(0|n)$.

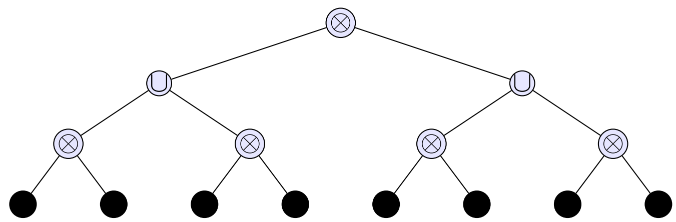

FIGURE 3. $T_G(2,2,0|0,0,2)$.

The following result is due [8].

Lemma 5. *Let G be a connected r-regular graph on n vertices for which the $\mathbf{A}$-eigenvalues are given by $\lambda_1 = r > \lambda_2 \geq \ldots \geq \lambda_n$. Then the $\mathbf{L}$-eigenvalues of G are $r - \lambda_n \geq r - \lambda_{n-1} \geq \ldots \geq r - \lambda_2 > r - \lambda_1 = 0$.*

Definition 1. *Let $a = (a_1, a_2, \ldots, a_n)$ be a fixed sequence of positive integers. We define the following parameter*

$$\gamma_{n,l} = \begin{cases} a_n a_{n-1} a_{n-2} \ldots a_l & if \quad 1 \leq l \leq n-1 \\ a_n & if \quad l = n. \end{cases}$$

The next result, $-(x_s^i)^m$ denotes the $\mathbf{A}$-eigenvalue of a cograph G with multiplicity m, produced by its cotree T_G at level $1 \leq i \leq s-1$. For more details, see [4]. From Lemma 5 follows:

Corollary 1. *Let G be a r-regular cograph on n vertices having balanced cotree T_G. Then, the $\mathbf{L}$-eigenvalues of G are given by*

(i) $\{(r + x_s^i)^{a_1 \ldots (a_i - 1)} \;\; for \;\; 1 \leq i \leq s-1, (r)^{a_1 \ldots (a_s - 1)}, n\}$, *if s is even, where*

$$\begin{cases} x_s^i = \sum_{k=1}^{s-i} \gamma_{s,i+k}(-1)^k & if \ i \ is \ even, \\ x_s^i = \sum_{k=1}^{s-i} \gamma_{s,i+k}(-1)^{k+1} & if \ i \ is \ odd. \end{cases}$$

(ii) $\{(r + x_s^i)^{a_1 \ldots (a_i - 1)} \;\; for \;\; 1 \leq i \leq s-1, (r+1)^{a_1 \ldots (a_s - 1)}, n\}$, *if s is odd, where*

$$\begin{cases} x_s^i = \sum_{k=1}^{s-i} \gamma_{s,i+k}(-1)^k + 1 & if \ i \ is \ even, \\ x_s^i = \sum_{k=1}^{s-i} \gamma_{s,i+k}(-1)^{k+1} + 1 & if \ i \ is \ odd. \end{cases}$$

The next result, presents an alternative representation of x_s^i, given by Corollary 1, for $1 \leq i \leq s-1$.

Lemma 6. *Let G be a r-regular cograph on n vertices having balanced cotree T_G. Then*

(i) $x_s^{s-1} = -a_s + 1$, *if s is odd , and for $j = 2, \ldots, s-1$*

$$x_s^{s-j} = (-1)^j a_s a_{s-1} \ldots a_{s-(j-1)} + x_s^{s-(j-1)}.$$

(ii) $x_s^{s-1} = a_s + 1$, *if s is even , and for $j = 2, \ldots, s-1$*

$$x_s^{s-j} = (-1)^{j+1} a_s a_{s-1} \ldots a_{s-(j-1)} + x_s^{s-(j-1)}.$$

Proof. We will prove by induction on j. We suppose s is odd. Using Corollary 1, since $s-1$ is even, we have that

$$x_s^{s-1} = \sum_{k=1}^{s-(s-1)} \gamma_{s,s-1+k}(-1)^k + 1 = -a_s + 1.$$

Applying Corollary 1 again, for $i = s-2$, follows

$$x_s^{s-2} = \sum_{k=1}^{2} \gamma_{s,s-2+k}(-1)^{k+2} + 1 = a_s a_{s-1} + x_s$$

$$x_s^{s-2} = (-1)^j a_s a_{s-1} + x_s^{s-1}$$

which proves the basis of our induction.

Now, we suppose that

$$x_s^{s-j} = (-1)^j a_s \dots a_{s-(j-1)} + x_s^{s-(j-1)}$$

and we want to prove that

$$x_s^{s-(j+1)} = (-1)^{j+1} a_s \dots a_{s-j} + x_s^{s-j}.$$

By Corollary 1, we have that

$$x_s^{s-(j+1)} = \sum_{k=1}^{j+1} \gamma_{s,s-(j+1)+k}(-1)^{k+1} + 1$$

$$\gamma_{s,s-j}(-1)^2 + \sum_{k=2}^{j+1} \gamma_{s,s-(j+1)+k}(-1)^{k+1} + 1$$

but, s and $s-(j+1)$ are odd numbers, then $(j+1)$ is an even number, therefore

$$x_s^{s-(j+1)} = (-1)^{j+1} a_s \dots a_{s-j} + \sum_{k=2}^{j+1} \gamma_{s,s-(j+1)+k}(-1)^{k+1} + 1.$$

Now, we want to show that

$$\sum_{k=2}^{j+1} \gamma_{s,s-(j+1)+k}(-1)^{k+1} + 1 = x_s^{s-j} \tag{3}$$

Since s and j are odd numbers, we have that $(s-j)$ is even, and

$$x_s^{s-j} = \sum_{k=1}^{j} \gamma_{s,s-j+k}(-1)^k + 1.$$

Performing the following change of variable $p = k+1$, follows

$$x_s^{s-j} = \sum_{p=2}^{j+1} \gamma_{s,s-(j+1)+p}(-1)^{p-1} + 1$$

$$= \sum_{p=2}^{j+1} \gamma_{s,s-(j+1)+p}(-1)^{p+1} + 1$$

which proves (3). Therefore

$$x_s^{s-(j+1)} = (-1)^{j+1} a_s \dots a_{s-j} + x_s^{s-j}$$

as desired. $\square$

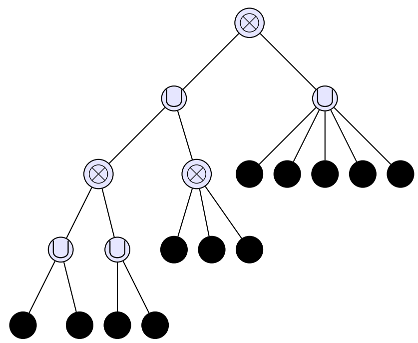

FIGURE 4. T_G corresponding to 7-regular cograph

3.2. **Non-balanced cotrees.** It is known that cographs having balanced cotree corresponds to r-regular graphs. However, not every r-regular cograph has balanced cotree. For example, the cotree of Figure 4 corresponds to 7-regular cograph, but it is obvious no balanced.

Let G be a cograph and T_G its cotree. We say T_G is a caterpillar if every interior vertex on T_G has exactly one interior vertex as an sucessor immediately. As illustration, on the left of Figure 5 shows a caterpillar cotree T_G.

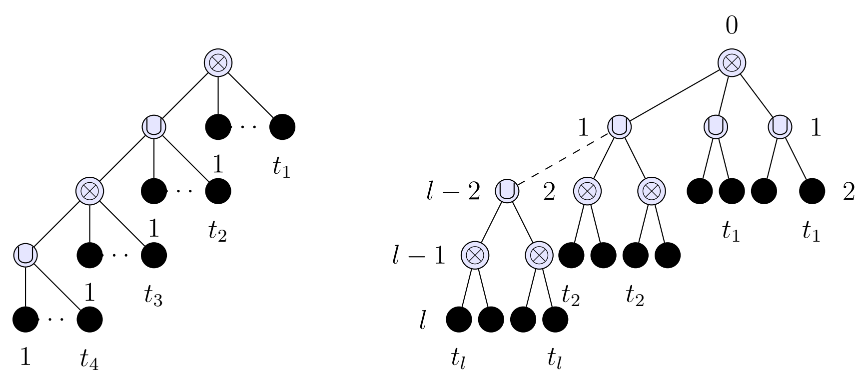

FIGURE 5. A caterpillar cotree and caterpillar having attached balanced cotrees

The following result presents the cotree's representation correspondent to a r-regular cograph.

Theorem 2. *Let G be a r-regular cograph with cotree T_G. If t_i are leaves of an interior vertex $w_i \in T_G$ then w_i is a terminal vertex. Furthermore, if $G = G_{n_1} \otimes G_{n_2}$, then $T_{G_{n_1}}$ and $T_{G_{n_2}}$ are either*

(1) a balanced cotree;

(2) a caterpillar having attached balanced cotrees (see, on the right of Figure 5).

Proof. Let G be a r-regular cograph with cotree T_G. If T_G is a balanced cotree the result holds. Now, we suppose that T_G is not a balanced cotree and let w_i be a non-terminal vertex having t_i leaves. If $w_i = \cup$-type there is an interior vertex $w'_{i+1} = \otimes$-type as successor. If t'_{i+1} are leaves below to w'_{i+1}, we have the degree of t'_{i+1} are greater than the degree of t_i, since any leaf in T_G which is adjacent to t_i must be adjacent to t'_{i+1}, which is a contradiction. If $w_i = \otimes$-type we can use a similar argument.

Now, we will prove the second statement. If T_G is not a balanced cotree we have terminal vertices in T_G have leaves in distinct levels. Since any path of alternating interior vertices containing no interior vertices as successor does not correspond to regular cograph, follows any interior vertex on the caterpillar of T_G must has an interior vertex as successor which is either a terminal vertex or a balanced cotrees. $\square$

Theorem 3. *Let G be a cograph on n vertices with cotree T_G. Let w be an interior vertex having t leaves and immediate sucessors $\{w_1, \ldots, w_j\}$, where each w_i has t_i leaves below. If there is t' leaves which are adjacent to w, then*

(i) *t' is a Laplacian eigenvalue with multiplicity $t + j - 1$, if $w = \cup$-type.*
(ii) *$t+t'+\sum_{i=1}^{j} t_i$ is a Laplacian eigenvalue with multiplicity $t+j-1$, if $w = \otimes$-type.*

Proof. For item [(i)], let $\cup$-type be an interior vertex having t leaves and immediate sucessors $\{w_1, \ldots, w_j\}$ where each w_i has t_i leaves below. If there is t' leaves which are adjacent to $\cup$-type, we can say the t leaves have degree t', while the leaves below of each w_i have degree $\delta(v_i) + t'$. To justify the $t + j - 1$ permanent zero values, we note to execute Diagonalization $(T_G, x = -t')$ with initial value $\delta(v_i) + t'$ is equivalent to execute Diagonalization $(T_G, x = 0)$ with value initial $\delta(v_i)$. By Lemma 4, follows $t + j$ zeros are assigned for the duplicate vertices of w. Applying **subcase 2a** , follows that $t + j - 1$ permanent zeros values are assigned.

Now, the second item [(ii)]. We assume G has order n, such that $n = p+t'+t+\sum_{i=1}^{j} t_i$. Consider the complement graph $\overline{G}$ and its cotree $T_{\overline{G}}$. It is easy to see that $t + \sum_{i=1}^{j} t_i$ vertices in $T_{\overline{G}}$ are adjacent to p leaves. By item [(i)] we have p is a Laplacian eigenvalue of $\overline{G}$ with multiplicity $t + j - 1$. Therefore $n - p = t + t' + \sum_{i=1}^{j} t_i$ is a Laplacian eigenvalue of G, as desired. $\square$

Theorem 4. *Let G be a r-regular cograph on n vertices having non-balanced cotree T_G. If $T_{G_1}, \ldots, T_{G_k}$ are the cotrees of degrees r_i $(i = 1, \ldots, k)$ attached in a caterpillar $\{w_1, w_2, \ldots, w_m\}$ of T_G. Then the* **L**-*eigenvalues non zero of G are given by*

(i) **L**-*eigenvalues of Corollary 1 plus t_i, if T_{G_i} is a balanced cotree adjacent to t_i leaves.*
(ii) $r_i + t_i + 1$ and $r_i + t_i - (b_i - 1)$ with multiplicity $b_i - 1$ and the number of T_{G_i} minus one, if T_{G_i} is a terminal vertex of $\otimes$-type having b_i leaves and adjacent to t_i leaves.
(iii) $r_i + t_i$ and $r_i + t_i + b_i$ with multiplicity $b_i - 1$ and the number of T_{G_i} minus one, if T_{G_i} is a terminal vertex of $\cup$-type having b_i leaves and adjacent to t_i leaves.
(iv) $n_i + n_j + t_i$, if w_i $(i = 1, \ldots, m)$ is of $\otimes$-type having cotrees of order n_i and n_j attached and adjacent to t_i leaves.
(v) t_i, if w_i $(i = 1, \ldots, m)$ is of $\cup$-type having cotrees attached and adjacent to t_i leaves.

Proof. Let G be a r-regular cograph having non-balanced cotree T_G with $T_{G_1}, \ldots, T_{G_k}$ cotrees of degrees r_i $(i = 1, \ldots, k)$ attached in a caterpillar $\{w_1, w_2, \ldots, w_m\}$ of T_G.

If T_{G_i} is a balanced cotree attached in an interior vertex $w_i \in T_G$ of degree r_i and adjacent to t_i leaves, we note that Diagonalization $(T_G, x = -t_i)$ with $\delta_i = r_i + t_i$ is equivalent to execute Diagonalization$(T_{G_i}, x = 0)$. Since 0 is a Laplacian eigenvalue of T_{G_i} follows t_i is a Laplacian eigenvalue of T_G. Repeating this procedure for the remaining Laplacian eigenvalues of T_{G_i}, by taking Diagonalization $(T_G, x = -t_i - x_s^i)$ and $\delta_i = r_i + t_i$, follows the item (i).

Now, let T_{G_i} and T_{G_j} be cotrees attached in an interior vertex w_i of a caterpillar of T_G, where T_{G_i} is a terminal vertex. If T_{G_i} is a terminal vertex of $\otimes$-type (respect. $\cup$-type) of degree r_i having b_i leaves and adjacent to t_i leaves, its Laplacian eigenvalue

is r_i+t_i+1 (respect. r_i+t_i) with multiplicity b_i-1, according to Lemma 1. For the remaining eigenvalues, we consider two cases:
Case 1: If $w_i=\cup$. According to Lemma 4, both cotrees T_{G_i} and T_{G_j} have assigned a value zero in the last iteration. Since T_{G_i} is terminal vertex with b_i leaves and adjacent to t_i leaves, by Lemma 2, follows $r_i+t_i+x=b_i-1$ and therefore $-x=r_i+t_i-(b_i-1)$ is a Laplacian eigenvalue common of T_{G_i} and T_{G_j}.
Case 2: If $w_i=\otimes$. We have cotrees T_{G_i} and T_{G_j} have received values -1 in the last iteration. Since T_{G_i} is terminal vertex with b_i leaves and adjacent to t_i leaves, by Lemma 3, follows $r_i+t_i+x=-b_i$ and therefore $-x=r_i+t_i+b_i$ is a Laplacian eigenvalue common of T_{G_i} and T_{G_j}. This proof the items (ii) and (iii).

The remaining items follow directly by Theorem 3. □

Example 1. *We consider the* 7*-regular cograph with non-balanced cotree* T_G *of Figure 4 with* $Spect_L(G)=\{12,9,8^2,7^6,5,0\}$.

We have T_{G_1} *and* T_{G_2} *are terminal vertices and* T_{G_3} *is a balanced cotree attached in the caterpillar* $\{w_1,w_2\}$ *of* T_G.

For cotree T_{G_1} *of* $\cup$*-type of degree* $r_1=0$ *with* $b_1=5$ *and* $t_1=7$*, follows by item* (iii), 7 *is a Laplacian eigenvalue with multiplicity 4, while* 12 *is a Laplacian eigenvalue with multiplicity 0.*

For cotree T_{G_2} *of* $\otimes$*-type of degree* $r_2=2$ *with* $b_2=3$ *and* $t_2=5$*, follows by item* (ii), 8 *is a Laplacian eigenvalue with multiplicity 2, while* 9 *is a Laplacian eigenvalue with multiplicity 0.*

For balanced cotree T_{G_3} *of degree* $r_3=2$ *and* $t_3=5$*, follows by item* (i) $4+5, 2+5, 2+5$ *are Laplacian eigenvalues.*

Finally, since T_G *has a caterpillar of depth 2, by items* (iv) *and* (v)*, we have* $3+4+5$ *and* 5 *are Laplacian eigenvalues.*

4. Regular cographs are L-DS

In this section, we prove that regular cographs are L-DS. We first prove no two r-regular cographs are L-cospectral.

Lemma 7. *Let* G *and* H *be two* r*-regular cographs on* n *vertices having balanced cotrees*

$$T_G(a_1,a_2\ldots,a_{s-1},0|0,\ldots,a_s) \tag{4}$$

and

$$T_H(b_1,b_2,\ldots,b_{s'-1},0|0,\ldots,b_{s'}) \tag{5}$$

respectively. If G *and* H *are* L*-cospectral then* $s=s'$.

Proof. Let G and H be two r-regular cographs on n vertices having balanced cotrees given by equations (4) and (5), respectively. It is sufficient to show that each level of cotree T_G produces a distinct Laplacian eigenvalue.

Suppose that $x_s^{s-(i+p)}=x_s^{s-i}$. By Lemma 6, we have

$$(-1)^{i+p}a_s\ldots a_{s-(i+p-1)}+\ldots+(-1)^{i+1}a_s\ldots a_{s-i}=0.$$

Dividing the last equation by $(-1)^{i+1}a_s\ldots a_{s-i}$, we have that

$$\frac{(-1)^{i+p}}{(-1)^{i+1}}a_{s-(i+1)}\ldots a_{s-(i+p-1)}+\ldots+\frac{(-1)^{i+2}}{(-1)^{i+2}}a_{s-(i+1)}+1=0.$$

The last equation implies

$$(-1)^{p-1}a_{s-(i+1)}\ldots a_{s-(i+p-1)}+\ldots+(-1)^2a_{s-(i+1)}\ldots a_{s-(i+1)}=-1.$$

Since $a_{s-(i+1)}$ is a common term of equation above, we have

$$a_{s-(i+1)} \cdot ((-1)^{p-1} a_{s-(i+2)} \dots a_{s-(i+p-1)} + \dots + (-1)^2 a_{s-(i+2)} + 1) = -1 \tag{6}$$

The equation (6) means we have a product of two integers numbers equal to -1. It implies that $a_{s-(i+1)} = 1$, what is a contradiction, since $a_k \geq 2$ for any k. $\square$

Theorem 5. *Let G and H be two r-regular cographs on n vertices having balanced cotrees T_G and T_H, respectively. If G and H are L-cospectral graphs then $G \cong H$.*

Proof. Let G and H be two r-regular cographs on n vertices having balanced cotrees $T_G(a_1, \dots, a_{s-1}, 0|0, \dots, a_s)$ and $T_H(b_1, \dots, b_{s-1}, 0|0, \dots, b_s)$. We claim that $a_i = b_i$, for $1 \leq i \leq s$. We proceed by induction on s.

For $s = 1$, since their complements $\overline{G}$ and $\overline{H}$ are also L-cospectral graphs, implies that number of components of $\overline{G}$ and $\overline{H}$ is the same. Therefore $a_1 = b_1$.

Now, we assume that for any two r-regular cograph H with balanced cotree T_H having the same Laplacian spectrum of a r-regular cograph G with balanced cotree T_G with depth less than s are isomorphic. Let G and H be a r-regular cographs on n vertices having balanced cotrees T_G and T_H with depth s. We note the complements $\overline{G}$ and $\overline{H}$ must be regular with same number of components, follows each component of $\overline{G}$ with $a_2 \dots a_s$ vertices and each component of $\overline{H}$ with $b_2 \dots b_s$ vertices have the same degree and same number of vertices. Since each component corresponds to balanced cotree with depth less than s, by induction follows they are isomorphic. Therefore, then $G \cong H$. $\square$

For the next technical result, G_t denotes a graph on t vertices.

Lemma 8. *Let G and H be two L-cospectral cographs on n vertices given by*

$$G = (G_{t_1} \cup G_{t_2} \cup \dots \cup G_{t_r}) \otimes G_{n_2} \tag{7}$$

$$H = (H_{t'_1} \cup H_{t'_2} \cup \dots \cup H_{t'_s}) \otimes H_{n'_2} \tag{8}$$

with $n_2 < \sum_{i=1}^r t_i$ and $n'_2 < \sum_{i=1}^r t'_i$. Then

(i) $n_2 = n'_2$ *and* $r = s$.
(ii) $\sum_{i=1}^r t_i = \sum_{i=1}^r t'_i$
(iii) $t_i = t'_i$ *for* $i = 1, \dots, r$.

Proof. Let G and H be two L-cospectral cographs on n vertices given by equations (7) and (8). Verifying the item (i). Since the second smallest Laplacian eigenvalue of a cograph corresponds to its vertex connectivity (see [1]) follows $\mu_2(G) = n_2 = n'_2 = \mu_2(H)$. To verify the second equality, we note n_2 is a Laplacian eigenvalue of G and H. Taking into account its multiplicity is equal to the number of componentes minus one on the left of equations (7) and (8), we have $r = s$. The item (ii) follows from item (i).

For verifying the item (iii) we consider the partial cotrees T_G and T_H at level three. By Theorem 3 (item (ii)), we have the following Laplacian eigenvalues:

$$t_1 + n_2, \dots, t_r + n_2, t'_1 + n_2, \dots, t'_r + n_2 \tag{9}$$

Since these are the maximum values obtained by an interior vertex $w_i = \otimes$-type in T_G (respect. $w'_i = \otimes$type in T_H) except to the cotree's root, by (9) we have $t_i = t'_i$, for $i = 1, \dots, r$, as desired. $\square$

Theorem 6. *No two r-regular cographs are L-cospectral.*

Proof. Let G and H be two r-regular cographs on n vertices which are L-cospectral with cotrees T_G and T_H, respectively. By Theorem 5, it is sufficient to verify the case of both cotrees T_G and T_H are non-balanced cotrees.

Assuming that G and H are connected graphs, we have

$$G = (G_{\sum t_i} \cup G_{t_2}) \otimes G_{t_1} \tag{10}$$

and

$$H = (H_{\sum t'_i} \cup H_{t'_2}) \otimes H_{t'_1} \tag{11}$$

with partial cotrees illustrated in the Figure 6. By Lemma 8, $t_1 = t'_1, t_2 = t'_2$ and $\sum t_i = \sum t'_i$. Let's denote by $T_G(t_1)$ and $T_H(t_1)$ the balanced cotrees attached in the cotree's root of T_G and T_H, respectively. We claim $T_G(t_1) \cong T_H(t_1)$.

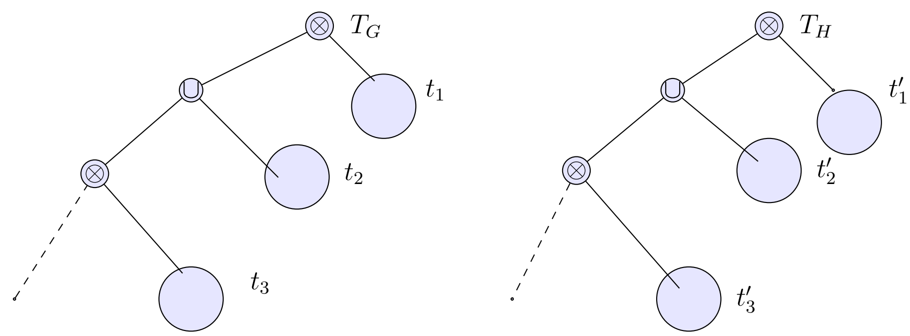

FIGURE 6. The partial cotrees T_G and T_H.

If $T_G(t_1)$ and $T_H(t_1)$ are t_1K_1 we are done. Now, we assume the cotrees have depth greater than one and $a_1a_2\ldots a_s = b_1b_2\ldots b_s = t_1$ leaves. By Theorem 3, $n - t_1$ is a common Laplacian eigenvalue of G and H with multiplicity $a_1 - 1$ and $b_1 - 1$. Therefore, thus $a_1 = b_1$. For the next level, we have $n - t_1 + (a_2 \ldots a_s)$ is a common Laplacian eigenvalue with multiplicity $a_2 - 1$ and $b_2 - 1$, which implies that $a_2 = b_2$. Repeating this procedure so on we have $T_G(t_1) \cong T_H(t_1)$.

From this, follows that $G_{\sum t_i} \cup G_{t_2}$ and $H_{\sum t_i} \cup H_{t_2}$ are L-cospectral cographs. By similar argument, we prove that $G_{t_2} \cong H_{t_2}$. Continuing this process, we will achieve either two balanced cotrees or a disjoint union of balanced cotrees having same order and degree. By Theorem 5, follows the result as desired. □

Lemma 9. *Let G_{n_1} and G_{n_2} be two graphs on n_1 and n_2 vertices, respectively, with $n_1 < n_2$. If G_{n_2} is not a connected graph then $G_{n_1} \otimes G_{n_2}$ has n_1 as Laplacian eigenvalue with multiplicity equals to the number of components of G_{n_2} minus one.*

Proof. Let $G_{n_1} \otimes G_{n_2}$ be a graph obtained from the join of G_{n_1} and G_{n_2} with $n_1 < n_2$. Since G_{n_2} is not a connected graph, we have n_1 is a Laplacian eigenvalue of $G_{n_1} \otimes G_{n_2}$. So, there are two ways to obtain n_1 :

- adding n_1 with zeros (number of components of G_{n_2})
- adding $n_1 - n_2$ with n_2.

Taking into account that $n_1 < n_2$, follows the multiplicity of n_1 corresponds to the number of components of G_{n_2} minus one, as desired. □

Theorem 7. *Let G be a r-regular cograph of order n. If H is a r-regular graph on n vertices having the same Laplacian spectrum of G then H is a cograph.*

Proof. We proceed by induction on r. If $r = 0$ we have H is nK_1. If $r = 1$ we have H is a disjoint union of copies of K_2. We suppose that any graph L-cospectral with G having degree less than r must be a cograph too. Now, let H be a r-regular graph having the same Laplacian spectrum with G. We consider the following cases.

Case 1: G and H are connected.

Since their complements graphs $\overline{G}$ and $\overline{H}$ also are L-cospectral and have degree less than r by induction follows the result.

Case 2: G and H are disconnected. Let $G = G_{n_1} \cup G_{n_2}$ and $H = H_{n_1} \cup H_{n_2}$ be two L-cospectral graphs. We suppose that G is a cograph and H is not a cograph. Then, we consider their complements

$$\overline{G} = \overline{G}_{n_1} \otimes \overline{G}_{n_2} \qquad \overline{H} = \overline{H}_{n_1} \otimes \overline{H}_{n_2} \tag{12}$$

Since $\overline{H}$ is a join and it is not a cograph, we can assume that the component $\overline{H}_{n_2}$ has P_4 as an induced subgraph and consider two subcases:

Subcase 2.1: $n_1 < n_2$.

Fact 1. *The multiplicity of Laplacian eigenvalue n_1 in $\overline{G}$ (resp. $\overline{H}$) is equal to the number of components of $\overline{G}_{n_2}$ (resp. $\overline{H}_{n_2}$) minus one.*

Since $n_1 < n_2$, this fact follows directly by Lemma 9.

Fact 2. *H_{n_2} is not a join.*

Suppose that H_{n_2} is a join. Since its complement is a disconnected graph then one of the components of $\overline{H}_{n_2}$ has P_4. By other hand each component of $\overline{H}_{n_2}$ has degree less than r and this contradicts the induction hypothesis.

Fact 3. *The multiplicity of n_1 in $\overline{H}$ differs from the multiplicity of n_1 in $\overline{G}$.*

Since H_{n_2} is not a join then its complement $\overline{H}_{n_2}$ is connected too. So, by Fact 1 we are done.

Subcase 2.2: $n_1 \geq n_2$.

We consider the L-cospectral graphs given by equation (12).

Fact 4. $\overline{G}_{n_1} \cong \overline{H}_{n_1}$.

Since $\overline{G}_{n_1}$ and $\overline{H}_{n_1}$ are $(r - n_2)$-regular cographs, we have their cotrees either correspond to disjoint union of balanced cotrees or non-balanced cotrees. Using a similar argument of the proof of Theorem 6, we have that $T_{\overline{G}_{n_1}} \cong T_{\overline{H}_{n_1}}$.

Fact 5. *$\overline{G}_{n_2}$ and $\overline{H}_{n_2}$ are L-cospectral graphs.*

Since the graphs given by equation (12) are L-cospectral by Fact 4 follows the result.

The Fact 5 implies that G_{n_2} and H_{n_2} are also L-cospectral graphs. By other hand G_{n_2} and H_{n_2} are regular graphs with degree less than r. Since H_{n_2} has P_4 as an induced subgraph this contradicts the induction hypothesis. □

Corollary 2. *Every r-regular cograph is determined by its Laplacian spectrum.*

5. Acknowledgements

This work was part of the Post-Doctoral studies of Fernando C. Tura, while visiting Georgia State University, on leave from UFSM and supported by CNPq Grant 200716/2022-0. Luis Emilio Allem is partially supported by grant FAPERGS(Proc. 21/2551-0002053-9) and Guantao Chen acknowledges partial support of NSF grant DMS-2154331.

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